BCS theory of hadronic matter at high densities

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Abstract

The equilibrium between the so-called 2SC and CFL phases of strange quark matter at high densities is investigated in the framework of a simple schematic model of the NJL type. Equal densities are assumed for quarks u, d and s. The 2SC phase is here described by a color-flavor symmetric state, in which the quark numbers are independent of the color-flavor combination. In the CFL phase the quark numbers depend on the color-flavor combination, that is, the number of quarks associated with the color-flavor combinations ur, dg, sb is different from the number of quarks associated with the color flavor combinations ug, ub, dr, db, sr, sg. We find that the 2SC phase is stable for a chemical potential μ below $\mu_c = 0.505$ GeV, while the CFL phase is stable above, the equilibrium pressure being $P_c = 0.003$ GeV^4. We have used a 3-momentum regularizing cutoff $\Lambda = 0.8$ GeV, which is somewhat larger than is usual in NJL type models. This should be adequate if the relevant chemical potential does not exceed 0.6 GeV.

1 Introduction

The color superconducting phase in quark matter is a topic of great current interest which has attracted the attention of many authors [1]. It is expected to be realized at very high densities, several times the equilibrium density of nuclear matter. For a recent review, see [2]. In the literature, the color superconducting phase is described by a BCS state which breaks the SU(3) color symmetry. This is unavoidable. However, in the conventional treatment of color superconductivity, a BCS state vector is used which gives rise to color-dependent quark numbers. This is undesirable and can be easily avoided [3]. Rudolf Peierls [4] classifies the BCS approach as a *symmetry breaking approximation* meaning that the relevant symmetry is not globally broken by the physical state itself but by the approximation used to take into account important correlations which are

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present in the physical state. In this note we make use of a version of the BCS formalism for quark matter [3] which, as far as possible, is compatible with color symmetry. Here, strange quark matter and some of its phases is investigated. In the conventional 2SC phase, 2 specified colors and 2 specified flavors are paired, the third color and flavor remaining as spectators. In addition, in the 2SC phase, beyond the chemical potential μ which fixes the total number of quarks, an extra Lagrange multiplier μ_8 is used to insure that the number of quarks is color independent, or, equivalently, to insure that $\langle \lambda_8 \rangle = 0$ [9, 10]. If an extra constraint is used, the thermodynamical potential is increased, the pressure is lowered and so the stability reduced, which is undesirable. We propose an alternative approach [3] in which the condition $\langle \lambda_8 \rangle = 0$ is automatically achieved without imposing it as an additional constraint as is done, for instance in [10]. Our approach has the advantage of being elementary and leads to several possible phases, namely, a color-flavor symmetric phase, a color symmetric flavor asymmetric phase, a color-flavor asymmetric phase. These phases are essentially equivalent to the 2SC phase except that they do not involve the Lagrange multiplier μ_8 and are therefore more stable. The so-called color flavor locked (CFL) phase [5] is also described. It is presently accepted that quark and gluon fields are the building blocks of hadronic matter, in the framework of quantum chromodynamics (QCD). A diversity of phases is expected at high densities: chiral-symmetry restoration, deconfinement and color-superconductivity. We focus on high densities for which chiral symmetry is expected to have been restored, that is, phases on which the color condensate $\langle \bar{q}q \rangle$ is expected to be no longer present. Since, due to the complexity of the theory, it is extremely difficult, if not impossible, to obtain exact results directly from QCD when perturbation theory cannot be applied, we follow the common practice and resort to an effective model of the Nambu-Jona-Lasinio (NJL) type containing a QCD inspired four-fermion interaction [6, 7, 8, 9, 11]. The present calculation is admittedly very schematic. In the end, we assume that the current masses of all quarks, including strange quarks, may be neglected. This rough approximation is acceptable if the chemical potential μ is high enough. The phase diagram of high density hadronic matter has been realistically treated in [10]. Although our discussion is based on a schematic and qualitative model, it suggests, for the critical chemical potential, a transition from the color symmetric flavor asymmetric phase to the CFL phase. In order to account for the tails of the occupation numbers, we use a 3-momentum regularizing cutoff $\Lambda = 0.8$ GeV, which is somewhat larger than is usual in NJL type models. We find that the color-flavor symmetrical phase is stable below the critical density $\rho_1 = 0.022$ GeV^3. The CFL phase is stable above the critical density $\rho_2 = 0.025$ GeV^3. Between ρ_1 and ρ_2 both phases co-exist.

The properties of quark matter at high densities may be of interest in connection with the description of the interior of neutron stars. In [5] it is suggested that it is made of strange matter, at least if the density is sufficiently high. The present results are consistent with this hypothesis.

2 Hamiltonian and thermodynamical potential

The thermodynamical potential is the Hamiltonian constrained by the Fermion number conservation. The chemical potential μ is the Lagrange multiplier which fixes the number of quarks. In the mean-field approximation, the thermodynamical potential operator reads,

$$\hat{K}_{MFA} = \int \mathrm{d}^3\mathbf{x} \left[\overline{\psi}(\mathbf{p} \cdot \boldsymbol{\gamma} + M - \mu\gamma_0)\psi + \frac{1}{2} \sum_{j,a \in \{2,5,7\}} \left((\Delta_{ja}^* \overline{\psi}^C i\gamma_5 \lambda_j^c \lambda_a^f \psi + h.c.) + \frac{|\Delta_{ja}|^2}{2G_c} \right) \right],$$

where

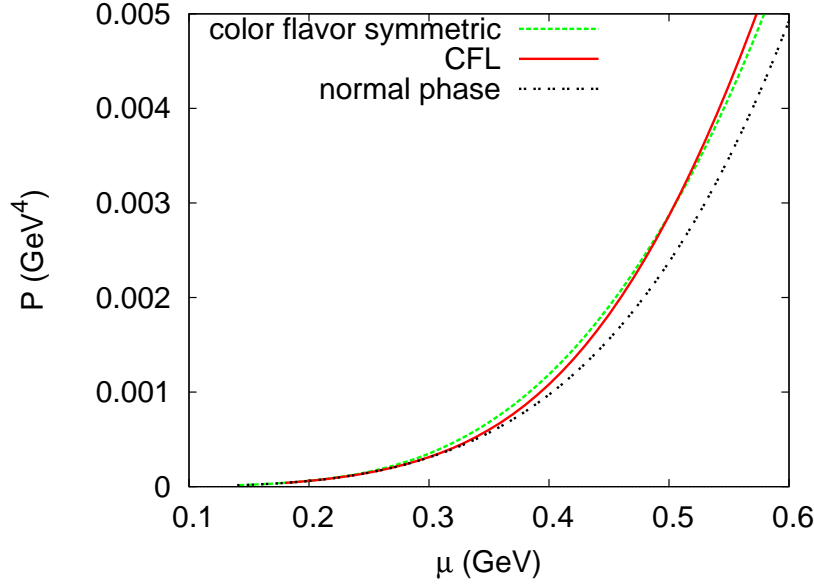$$\Delta_{ja} = -G_c \langle \overline{\psi}^C i\gamma_5 \lambda_j^c \lambda_a^f \psi \rangle.$$

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Figure 1: Transition from the color-flavor symmetric phase to the CFL phase. This figure shows that the color flavor symmetric phase is stable for μ below $\mu_c = 0.505$ GeV, and the CFL phase is stable for μ above that chemical potential, the equilibrium pressure being $P_c = 0.003$ GeV^4. The curves for the color symmetric flavor asymmetric and for the color-flavor asymmetric phases are superimposed with the curve for the color-flavor symmetric phase, which is shown.

Here, $\langle \overline{\psi}^C i\gamma_5 \lambda_j^c \lambda_a^f \psi \rangle$ denotes the expectation value of the diquark condensate $(\overline{\psi}^C i\gamma_5 \lambda_j^c \lambda_a^f \psi)$ in the BCS vacuum, and $\lambda_j^c, \lambda_a^f, j, a \in \{2, 5, 7\}$ are the antisymmetric Gell-Mann matrices, for color and flavor, respectively. The current quark mass is M and the chemical potential is μ. We wish to work in momentum space, so we need the expression in momentum space of the operators $(\overline{\psi}^C i\gamma_5 \lambda_j^c \lambda_a^f \psi)$ for $j, a \in \{2, 5, 7\}$. These quantities are associated with the $\overline{3}$ representations of color and flavor $SU(3)$. The mean field Hamiltonian density is expressed in terms of the diquark condensate. However, the mean field Hamiltonian itself involves the respective integral $\int \mathrm{d}^3\mathbf{x}(\overline{\psi}^C i\gamma_5 \lambda_j^c \lambda_a^f \psi)$. For $a = 2, j = 2$, we find

$$\frac{1}{V} \int \mathrm{d}^3\mathbf{x}(\overline{\psi}^C i\gamma_5 \lambda_2^c \lambda_2^f \psi) = \sum_p (c_{2dp} c_{1u\overline{p}} + c_{1up} c_{2d\overline{p}} - c_{2up} c_{1d\overline{p}} - c_{1dp} c_{2u\overline{p}})\eta_p$$

$$+ \sum_p (\tilde{c}_{2dp}^\dagger \tilde{c}_{1u\overline{p}}^\dagger + \tilde{c}_{1up}^\dagger \tilde{c}_{2d\overline{p}}^\dagger - \tilde{c}_{2up}^\dagger \tilde{c}_{1d\overline{p}}^\dagger - \tilde{c}_{1dp}^\dagger \tilde{c}_{2u\overline{p}}^\dagger)\eta_p, \quad \eta_p = -\eta_{\overline{p}},$$

with analogous expressions for the other choices of a, j. Here, c_{jbp} annihilates a quark with color j flavor b and momentum-helicity p, while $\tilde{c}_{jbp}^\dagger$ creates an anti-quark with color j flavor b and momentum-helicity p, and $\overline{p}$ is the state with the same helicity as p but opposite momentum. Since the Cooper instability occurs at the Fermi surface, we will ignore anti-quarks, in the sequel. Therefore, we will assume that in the mean-field

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approximation the thermodynamical potential reads simply,

$$
\begin{aligned}
\hat{K}_{MFA} &= \sum_{iap} \epsilon_p c_{iap}^\dagger c_{iap} \\
&+ \sum_{ijkabcp} (\Delta_{kc}^* \, c_{jbp} \, c_{ia\overline{p}} \, \epsilon_{ijk}\epsilon_{abc}\eta_p + \Delta_{kc} \, c_{iap}^\dagger \, c_{jb\overline{p}}^\dagger \, \epsilon_{ijk}\epsilon_{abc}\eta_p) + \frac{V}{2G_c} \sum_{jc} \Delta_{jc}\Delta_{jc}^*, \\
\Delta_{kc} &= \frac{G_c}{V} \sum_{ijabp} \langle c_{jbp} \, c_{ia\overline{p}} \rangle \, \epsilon_{ijk}\epsilon_{abc}\eta_p,
\end{aligned}
\tag{1}
$$

where $\epsilon_p = \sqrt{p^2 + M^2} - \mu$. It is convenient to define the effective gap as

$$
\Delta_{eff} = \sqrt{\frac{1}{9} \sum_{jc} \Delta_{jc}\Delta_{jc}^*} \; .
$$

We denote the color indices as $1, 2, 3$, meaning *red, green, blue*, and the flavor indices as a, b, c. Upon a color or flavor rotation the gaps Δ_{jc}, $j \in \{1, 2, 3\}$, $c \in \{u, d, s\}$, change but Δ_{eff} and the equation of state do not change.
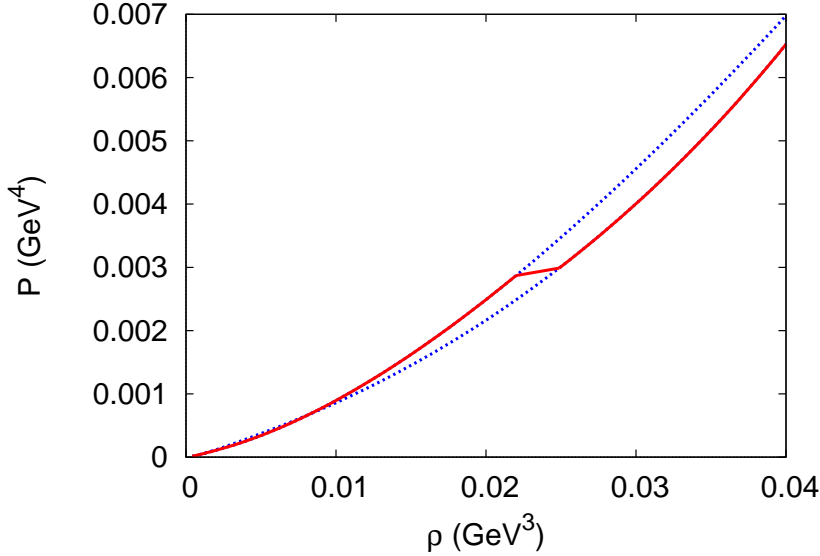

Figure 2: Pressure vs quark density. This figure illustrates the phase equilibrium between the CFL phase (at higher densities) and color-flavor symmetric phase (at lower densities). The dotted curve shows the un-physical continuation of the CFL curve (at lower densities) and of the color flavor symmetric curve (at higher densities). The two phases coexist for ρ between $\rho_1 = 0.022$ GeV^3 and $\rho_2 = 0.025$ GeV^3, the equilibrium pressure being $P_c = 0.003$ GeV^4. The curves for the color symmetric flavor asymmetric and for the color-flavor asymmetric phases are superimposed with the curve for the color-flavor symmetric phase, which is shown. The maximum density presented corresponds to 9.808 times the nuclear saturation density.

3 Color superconductivity

We consider the following super-conducting phases: 1) a color-flavor symmetric phase; 2) a color symmetric, flavor asymmetric phase; 3) a color-flavor asymmetric phase; 4) the so-called CFL phase [5]. The color-flavor symmetric phase is defined by the condition that the gaps Δ_{ia} are non-zero and equal, independently of color and flavor, that is, $\Delta_{ia} = \Delta \neq 0$, $i \in \{1,2,3\}$, $a \in \{u,d,s\}$. The color symmetric flavor asymmetric phase is defined by $\Delta_{iu} = \Delta_{id} = \Delta$, $\Delta_{is} = 0$, $i \in \{1,2,3\}$. The color and flavor asymmetric phase is defined by $\Delta_{1u} = \Delta_{2u} = \Delta_{1d} = \Delta_{2d} = \Delta$, $\Delta_{is} = \Delta_{3a} = 0$, $i \in \{1,2,3\}$, $a \in \{u,d,s\}$. In the CFL phase, by definition, the gaps satisfy $\Delta_{1u} = \Delta_{2d} = \Delta_{3s} \neq 0$, while $\Delta_{1d} = \Delta_{1s} = \Delta_{2u} = \Delta_{2s} = \Delta_{3u} = \Delta_{3d} = 0$.

3.1 Color-flavor symmetric phase

The BCS vacuum of the color-flavor symmetric phase is given by

$$|\Phi\rangle = e^{S}|\Phi_0\rangle, \quad |\Phi_0\rangle = \prod_{iap(\epsilon_p \leq 0)} c^{\dagger}_{iap}|0\rangle,$$

where

$$S = -\sum_{p(\epsilon_p > 0)} K_p \sum_{ijkabc} c^{\dagger}_{iap} c^{\dagger}_{jb\overline{p}} \epsilon_{ijk} \epsilon_{abc} \eta_p - \sum_{p(\epsilon_p \leq 0)} \widetilde{K}_p \sum_{ijkabc} c_{iap} c_{jb\overline{p}} \epsilon_{ijk} \epsilon_{abc} \eta_p.$$

It follows that the BCS vacuum satisfies

$$d_{iap}|\Phi\rangle = d_{ia\overline{p}}|\Phi\rangle = 0, \quad i \in \{1,2,3\}, a \in \{u,d,s\}$$

being, for $\epsilon_p = \sqrt{p^2 + M^2} - \mu > 0$,

$$d_{1up} = c_{1up} + K_p(c^{\dagger}_{2d\overline{p}} - c^{\dagger}_{2s\overline{p}} - c^{\dagger}_{3d\overline{p}} + c^{\dagger}_{3s\overline{p}})\eta_p, \quad \eta_{\overline{p}} = -\eta_p,$$

the other operators are obtained by cyclic permutations of the indices 1,2,3 and u,d,s, while for $\epsilon_p = \sqrt{p^2 + M^2} - \mu \leq 0$

$$d_{1up} = c^{\dagger}_{1up} - \widetilde{K}_p(c_{2d\overline{p}} - c_{2s\overline{p}} - c_{3d\overline{p}} + c_{3s\overline{p}})\eta_p,$$

plus cyclic permutations of the indices 1,2,3 and u,d,s. The operators d_{iap}, $d^{\dagger}_{iap}$ do not obey canonical anti-commutation relations, since, $\{d_{iap}, d^{\dagger}_{jbq}\} \neq \delta_{ij}\delta_{ab}\delta_{pq}$, although $\{d_{iap}, d_{jbq}\} = 0$. New operators f_{iap} are easily constructed as linear combinations of the operators d_{iap} which obey the canonical anti-commutation relations. These operators are not needed for the present purpose, so that their expressions are not presented. It is easily found that the optimal BCS vacuum leads to the gap equation

$$1 = \frac{2G}{V}\left(\sum_{p,\epsilon_p \leq 0} + \sum_{p,\epsilon_p > 0}\right)\frac{1}{\sqrt{\epsilon_p^2 + 9\Delta^2}},$$

and to the expression for the average thermodynamical potential (1), which reads

$$\langle \hat{K}_{MFA} \rangle = \sum_{p,\epsilon_p \leq 0}\left(7\epsilon_p - 2\sqrt{\epsilon_p^2 + 9\Delta^2}\right) + \sum_{p,\epsilon_p > 0}\left(2\epsilon_p - 2\sqrt{\epsilon_p^2 + 9\Delta^2}\right) + 9V\frac{\Delta^2}{2G_c}.$$

The equations of motion of the operators $c^{\dagger}_{iap}$ are straightforwardly obtained. As an example, the following commutation relation is presented,

$$[\hat{K}_{MFA}, c^{\dagger}_{1up}] = \epsilon_p c^{\dagger}_{1up} - \Delta(c_{2d\overline{p}} + c_{3s\overline{p}} - c_{2s\overline{p}} - c_{3d\overline{p}}).$$

Using these commutation relations, standard techniques allow the determination of linear combinations $f_p^\dagger = \sum(x_{ja}c_{jap}^\dagger + y_{ja}c_{ja\overline{p}})$ satisfying

$$[\hat{K}_{MFA}, f_p^\dagger] = \omega_p f_p^\dagger.$$

The diagonalization of the operator $\hat{K}_{MFA}$ is now easily achieved. One finds that the eigenvalues ω_p are ϵ_p (5-fold degenerate) and $\sqrt{\epsilon_p^2 + 9\Delta^2}$ (4-fold degenerate). Finally, for appropriate fermion operators $f_{kp}^\dagger$ we may write

$$\hat{K}_{MFA} = \sum_{j=1}^{5} |\epsilon_p| f_{jp}^\dagger f_{jp} + \sum_{j=6}^{9} \sqrt{\epsilon_p^2 + 9\Delta^2} f_{jp}^\dagger f_{jp} + \langle \hat{K}_{MFA} \rangle.$$

In the color-flavor symmetric case, we have $\Delta_{eff} = \Delta$. We observe that in the color-flavor symmetric phase the BCS quasi particles $f_{kp}^\dagger$ are not characterized by well defined quantum numbers. This is the price we have to pay for obtaining color-flavor indepen-dent quark numbers.
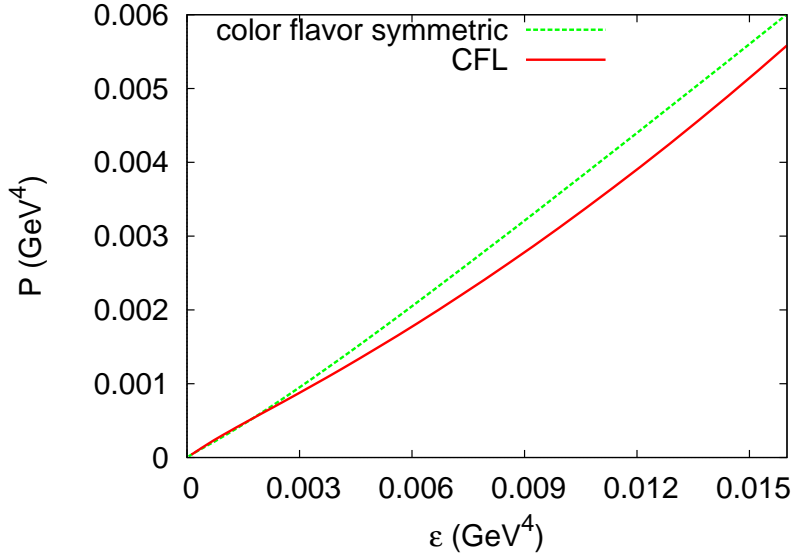

Figure 3: Pressure vs energy density. The red curve is physically meaningful for $P > 0.003$ GeV. The other curve is physically meaningful below that pressure. The curves for the color symmetric flavor asymmetric and for the color-flavor asymmetric phases are superimposed with the curve for the color-flavor symmetric phase, which is shown.

3.2   CFL phase
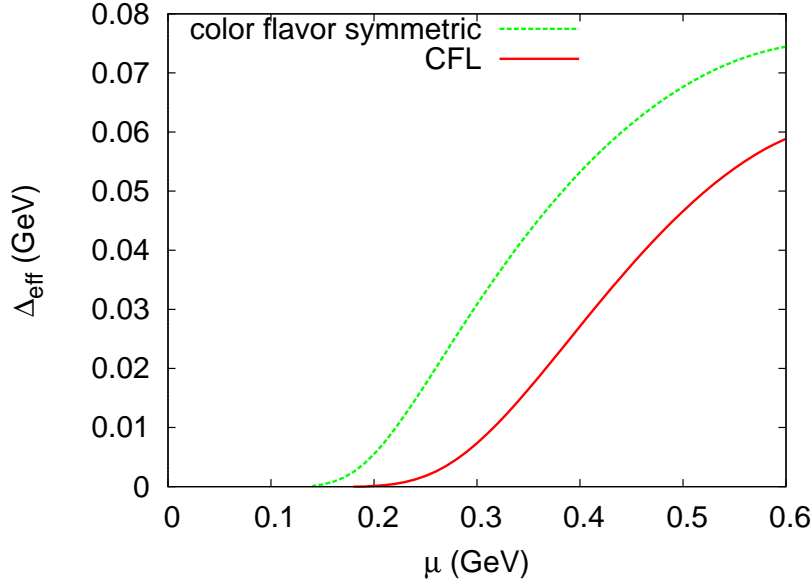
Next, we discuss the so-called CFL phase [5].

Figure 4: The efective gap Δ_{eff} for several phases. The curves for the color symmetric flavor asymmetric and for the color-flavor asymmetric phases are superimposed with the curve for the color-flavor symmetric phase, which is shown.

For a fixed p, let us consider the following effective Hamiltonian,

$$
\begin{aligned}
H_{eff} \;=\; & \epsilon \sum_{i=1}^{3} \sum_{a\in\{u,d,s\}} (c_{iap}^{\dagger} c_{iap} + c_{ia\overline{p}}^{\dagger} c_{ia\overline{p}}) \\
& + \Delta \sum_{i,j,k=1}^{3} \sum_{a,b\in\{u,d,s\}} (\, c_{jbp}\, c_{ia\overline{p}} \;+\; c_{iap}^{\dagger}\, c_{jb\overline{p}}^{\dagger}\,)\, \epsilon_{ijk}\epsilon_{abc_k},
\end{aligned}
$$

where $c_1 = u$, $c_2 = d$, $c_3 = s$, which involves only the di-quark condensates $(c_{2dp}c_{1u\overline{p}} + c_{1up}c_{2d\overline{p}} - c_{2up}c_{1d\overline{p}} - c_{1dp}c_{2u\overline{p}})$, $(c_{3sp}c_{2d\overline{p}} + c_{2dp}c_{3s\overline{p}} - c_{3dp}c_{2s\overline{p}} - c_{2sp}c_{3d\overline{p}})$, $(c_{1up}c_{3s\overline{p}} + c_{3sp}c_{1u\overline{p}} - c_{1sp}c_{3u\overline{p}} - c_{3up}c_{1s\overline{p}})$, and their hermitian adjoints. Having in mind the equations of motion of the operators $c_{iap}^{\dagger}$, exemplified by

$$
[H_{eff}, c_{1up}^{\dagger}] = \epsilon c_{1up}^{\dagger} - \Delta(c_{2d\overline{p}} + c_{3s\overline{p}}),
$$
$$
[H_{eff}, c_{1dp}^{\dagger}] = \epsilon c_{1dp}^{\dagger} + \Delta(c_{2u\overline{p}}), \quad [H_{eff}, c_{1sp}^{\dagger}] = \epsilon c_{1sp}^{\dagger} + \Delta(c_{3u\overline{p}}),
$$

we easily show that H_{eff} is diagonalized by the BCS transformation

$$
d_{jp}^{\dagger} = \sum_{i=1}^{3} (x_i^{(j)} c_{ia_i p}^{\dagger} + y_i^{(j)} c_{ia_i \overline{p}}), \quad d_{j\overline{p}}^{\dagger} = \sum_{i=1}^{3} (x_i^{(j)} c_{ia_i \overline{p}}^{\dagger} - y_i^{(j)} c_{ia_i p}), \quad j = 1, 2, 3, \quad (2)
$$

being $a_1 = u$, $a_2 = d$, $a_3 = s$, and

$$x_1^{(1)} = x_2^{(1)} = x_3^{(1)} = \sqrt{\frac{\epsilon + \sqrt{\epsilon^2 + 4\Delta^2}}{6\sqrt{\epsilon^2 + 4\Delta^2}}}, \quad y_1^{(1)} = y_2^{(1)} = y_3^{(1)} = \sqrt{\frac{-\epsilon + \sqrt{\epsilon^2 + 4\Delta^2}}{6\sqrt{\epsilon^2 + 4\Delta^2}}},$$

$$x_1^{(2)} = -x_2^{(2)} = \frac{1}{2}\sqrt{\frac{\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{\sqrt{\epsilon^2 + \Delta^2}}}, \quad x_1^{(3)} = x_2^{(3)} = -\frac{1}{2}x_3^{(3)} = \frac{1}{2}\sqrt{\frac{\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{3\sqrt{\epsilon^2 + \Delta^2}}},$$

$$-y_1^{(2)} = y_2^{(2)} = \frac{1}{2}\sqrt{\frac{-\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{\sqrt{\epsilon^2 + \Delta^2}}}, \quad -y_1^{(3)} = -y_2^{(3)} = \frac{1}{2}y_3^{(3)} = \frac{1}{2}\sqrt{\frac{-\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{3\sqrt{\epsilon^2 + \Delta^2}}},$$

$$x_3^{(2)} = y_3^{(2)} = 0.$$

The relations (2) are completed by

$$d_{4p}^\dagger = X c_{1dp}^\dagger + Y c_{2u\overline{p}}, \quad d_{5p}^\dagger = X c_{2up}^\dagger + Y c_{1d\overline{p}},$$

$$d_{4\overline{p}}\dagger = X c_{1d\overline{p}}^\dagger - Y c_{2up}, \quad d_{5\overline{p}}\dagger = X c_{2u\overline{p}}^\dagger - Y c_{1dp},$$

plus analogous relations for the remaining color-flavor combinations, being

$$X = \sqrt{\frac{\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{2\sqrt{\epsilon^2 + \Delta^2}}}, \quad Y = \sqrt{\frac{-\epsilon + \sqrt{\epsilon^2 + \Delta^2}}{2\sqrt{\epsilon^2 + \Delta^2}}}.$$

Its eigenvalues are, $\sqrt{\epsilon^2 + 4\Delta^2}$, (2-fold degenerate) and $\sqrt{\epsilon^2 + \Delta^2}$ (16-fold degenerate), so that

$$\begin{aligned} H_{eff} =& \sqrt{\epsilon^2 + 4\Delta^2}(d_{1p}^\dagger d_{1p} + d_{1\overline{p}}^\dagger d_{1\overline{p}}) + \sqrt{\epsilon^2 + \Delta^2} \sum_{j=2}^{9}(d_{jp}^\dagger d_{jp} + d_{j\overline{p}}^\dagger d_{j\overline{p}}) \\ &+ 9\epsilon - 8\sqrt{\epsilon^2 + \Delta^2} - \sqrt{\epsilon^2 + 4\Delta^2}. \end{aligned}$$

We postulate that the CFL phase is the vacuum of the quasi-particles defined by (2) for p such that $\epsilon_p > 0$ and by a similar transformation, in which, however, the states $c_{jap}^\dagger$ are regarded as antipaticles, for p such that $\epsilon_p \leq 0$, that is

$$d_{jp} = \sum_{i=1}^{3}(x_i^{(j)} c_{ia_ip}^\dagger + y_i^{(j)} c_{ia_i\overline{p}}), \quad d_{j\overline{p}}^\dagger = \sum_{i=1}^{3}(x_i^{(j)} c_{ia_i\overline{p}}^\dagger - y_i^{(j)} c_{ia_ip}), \qquad (3)$$

The relations (3) are completed by

$$d_{4p} = X c_{1dp}^\dagger + Y c_{2u\overline{p}}, \quad d_{5p} = X c_{2up}^\dagger + Y c_{1d\overline{p}},$$

$$d_{4\overline{p}} = X c_{1d\overline{p}}^\dagger - Y c_{2up}, \quad d_{5\overline{p}} = X c_{2u\overline{p}}^\dagger - Y c_{1dp},$$

plus analogous relations for the remaining color-flavor combinations. For the gap equation in the CFL phase we obtain

$$1 = \frac{G}{3V} \left(\sum_{p,\epsilon_p \leq 0} + \sum_{p,\epsilon_p > 0} \right) \left(\frac{4}{\sqrt{\epsilon_p^2 + 4\Delta^2}} + \frac{8}{\sqrt{\epsilon_p^2 + \Delta^2}} \right)$$

and for the average thermodynamical potential we find

$$\begin{aligned} \langle \hat{K}_{MFA} \rangle =& \sum_{p,\epsilon_p \leq 0} \left(-\sqrt{\epsilon_p^2 + 4\Delta^2} - 8\sqrt{\epsilon_p^2 + \Delta^2} \right) \\ &+ \sum_{p,\epsilon_p > 0} \left(9\epsilon_p - \sqrt{\epsilon_p^2 + 4\Delta^2} - 8\sqrt{\epsilon_p^2 + \Delta^2} \right) + 3V \frac{\Delta^2}{2G_c}. \end{aligned}$$

We observe that the gap equation for the CFL phase is essentially different from the gap equations for all the other phases. In the CFL phase, $\Delta_{eff} = \Delta/\sqrt{3}$.

3.3 Color symmetric flavor asymmetric phase

In the case of the color symmetric flavor asymmetric phase, for $\epsilon_p = \sqrt{p^2 + M^2} - \mu \geq 0$, the BCS vacuum is annihilated by the operators

$$d_{1up} = c_{1up} + K_p(c_{2d\overline{p}}^\dagger - c_{3d\overline{p}}^\dagger), \quad d_{1dp} = c_{1dp} + K_p(c_{2u\overline{p}}^\dagger - c_{3u\overline{p}}^\dagger),$$

$$d_{2up} = c_{2up} + K_p(c_{3d\overline{p}}^\dagger - c_{1d\overline{p}}^\dagger), \quad d_{2dp} = c_{2dp} + K_p(c_{3u\overline{p}}^\dagger - c_{1u\overline{p}}^\dagger)$$

$$d_{3up} = c_{3up} + K_p(c_{1d\overline{p}}^\dagger - c_{2d\overline{p}}^\dagger), \quad d_{3dp} = c_{3dp} + K_p(c_{1u\overline{p}}^\dagger - c_{2u\overline{p}}^\dagger)$$

$$d_{isp} = c_{isp}, \quad i \in \{1, 2, 3\}$$

while for $\epsilon_p = \sqrt{p^2 + M^2} - \mu \leq 0$, the BCS vacuum is annihilated by

$$d_{1up} = c_{1up}^\dagger - \widetilde{K}_p(c_{2d\overline{p}} - c_{3d\overline{p}}), \quad d_{1dp} = c_{1dp}^\dagger - \widetilde{K}_p(c_{2u\overline{p}} - c_{3u\overline{p}}),$$

$$d_{2up} = c_{2up}^\dagger - \widetilde{K}_p(c_{3d\overline{p}} - c_{1d\overline{p}}), \quad d_{2dp} = c_{2dp}^\dagger - \widetilde{K}_p(c_{3u\overline{p}} - c_{1u\overline{p}})$$

$$d_{3up}^\dagger = c_{3up}^\dagger - \widetilde{K}_p(c_{1d\overline{p}} - c_{2d\overline{p}}), \quad d_{3dp}^\dagger = c_{3dp}^\dagger - \widetilde{K}_p(c_{1u\overline{p}} - c_{2u\overline{p}})$$

$$d_{isp} = c_{isp}^\dagger, \quad i \in \{1, 2, 3\}.$$

In this case, the gap equation reads,

$$1 = \frac{2G}{V} \left(\sum_{p,\epsilon_p \leq 0} + \sum_{p,\epsilon_p > 0} \right) \frac{1}{\sqrt{\epsilon_p^2 + 3\Delta^2}}.$$

and the thermodynamical potential reads,

$$\langle \hat{K}_{MFA} \rangle = \sum_{p,\epsilon_p \leq 0} \left(7\epsilon_p - 2\sqrt{\epsilon_p^2 + 3\Delta^2} \right) + \sum_{p,\epsilon_p > 0} \left(2\epsilon_p - 2\sqrt{\epsilon_p^2 + 3\Delta^2} \right) + 3V \frac{\Delta^2}{2G_c}.$$

In the color symmetric flavor asymmetric phase, $\Delta_{eff} = \Delta/\sqrt{3}$.

3.4 Color and flavor asymmetric phase

In the case of the color and flavor asymmetric phase, for $\epsilon_p = \sqrt{p^2 + M^2} - \mu \geq 0$, the BCS vacuum is annihilated by the opertors

$$d_{1up} = c_{1up} + K_p c_{2d\overline{p}}^\dagger, \quad d_{1dp} = c_{1dp} - K_p c_{3u\overline{p}}^\dagger,$$

$$d_{2up} = c_{2up} - K_p c_{1d\overline{p}}^\dagger, \quad d_{2dp} = c_{2dp} + K_p c_{3u\overline{p}}^\dagger$$

$$d_{3up} = c_{3up}, \quad d_{3dp} = c_{3dp}, \quad d_{isp} = c_{isp}, \quad i \in \{1, 2, 3\}$$

while for $\epsilon_p = \sqrt{p^2 + M^2} - \mu \leq 0$, it is annihilated by

$$d_{1up} = c_{1up}^\dagger - \widetilde{K}_p c_{2d\overline{p}}, \quad d_{1dp} = c_{1dp}^\dagger - \widetilde{K}_p c_{2u\overline{p}}$$

$$d_{2up} = c_{2up}^\dagger + \widetilde{K}_p c_{1d\overline{p}}, \quad d_{2dp} = c_{2dp}^\dagger + \widetilde{K}_p c_{1u\overline{p}}$$

$$d_{3up} = c_{3up}^\dagger, \quad d_{3dp} = c_{3dp}^\dagger, \quad d_{isp} = c_{isp}^\dagger, \quad i \in \{1, 2, 3\}.$$

The gap equation reads

$$1 = \frac{2G}{V} \left(\sum_{p,\epsilon_p \leq 0} + \sum_{p,\epsilon_p > 0} \right) \frac{1}{\sqrt{\epsilon_p^2 + \Delta^2}}.$$
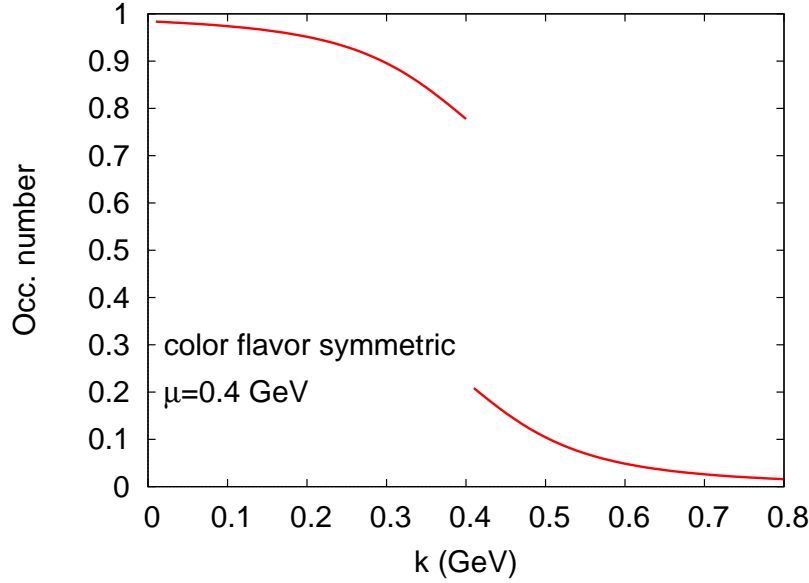
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Figure 5: Occupation factors for the color-flavor symmetric phase. They are independent of the color-flavor combination. For $\mu = 0.4$ GeV, $\Delta = 0.0532$ GeV. The color-flavor symmetric phase is stable for $\mu \leq 0.505$ GeV.

and the thermodynamical potential is given by

$$\langle \hat{K}_{MFA} \rangle = \sum_{p,\epsilon_p \leq 0} \left(7\epsilon_p - 2\sqrt{\epsilon_p^2 + \Delta^2} \right) + \sum_{p,\epsilon_p > 0} \left(2\epsilon_p - 2\sqrt{\epsilon_p^2 + \Delta^2} \right) + V\frac{\Delta^2}{2G_c}.$$

In the color and flavor asymmetric phase, $\Delta_{eff} = \Delta/3$.

If written in terms of Δ_{eff}, the thermodynamical potentials $\langle \hat{K}_{MFA} \rangle$ for the color-flavor symmetric phase, for the color symmetric flavor asymmetric phase and for the color-flavor asymmetric phase, coincide. These phases are essentially equivalent. The only difference is that in the color-flavor symmetric phase the occupation numbers are color-flavor independent and in the remaining phases they are not. The respective BCS vacua are related to each other by simple color-flavor rotations. To see that the color-flavor asymmetric phase is unitarily similar to the color-symmetric flavor-asymmetric phase, consider the matrix in $SU(\overline{3})$ which transforms the vector $(1, 0, 0)$ into $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. The operator induced by the same matrix transforms the color-flavor asymmetric BCS vacuum into the color-symmetric flavor-asymmetric BCS vacuum. The 2SC phase which is described in the literature (e. g., see [10]) differs from the color-flavor asymmetric phase here considered in using or not using the Lagrange multiplier μ_8 to implement the constraint $\langle \lambda_8 \rangle = 0$. The present work shows that this implementation should not be imposed. Although $n_1 = n_2 \neq n_3$, the color rotation mentioned above transforms the colors $1, 2, 3$ into new colors $1', 2', 3'$ such that $n_{1'} = n_{2'} = n_{3'}$. Nothing prevents us from regarding $1', 2', 3'$ as the physical colors and $1, 2, 3$ as auxiliary colors. Avoiding the constraint $\langle \lambda_8 \rangle = 0$ will have an effect on the critical point which is not irrelevant because, as fig. 1 shows, the P, μ curves have very close slopes.
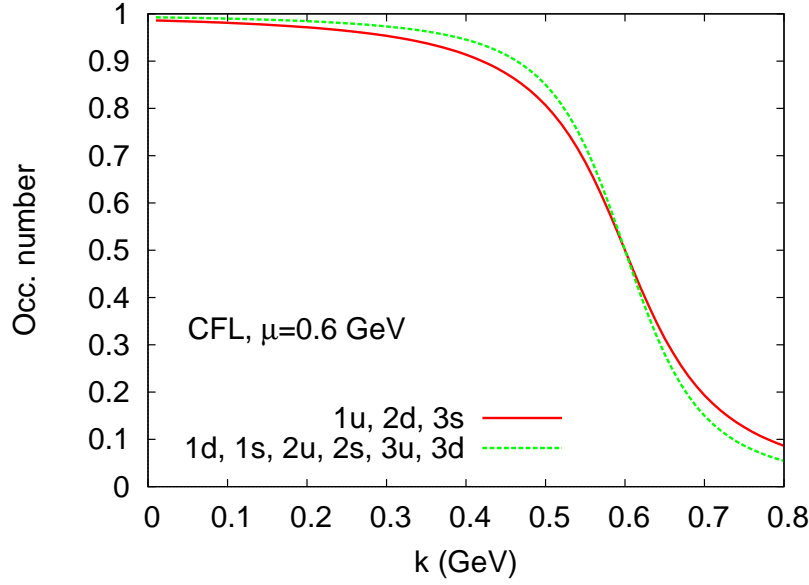
Figure 6: Occupation factors for the CFL phase. Full curve: (1u), (2d), (3s) color-flavor combinations. Dashed curve: remaining combinations. For $M = 0$, $\mu = 0.6$ GeV, $\Delta = 0.102$ GeV. The CFL phase is stable for $\mu \geq 0.505$ GeV.

4 Occupation numbers

It is clear that the curve 'pressure versus chemical potential' is the same for the phases color-flavor symmetric, color symmetric flavor asymmetric and color-flavor asymmetric. Only the color-flavor dependence of the occupation numbers distinguishes these phases.

For the color-flavor symmetric phase, we find:

$$\langle\Phi|c_{iap}^\dagger c_{iap}|\Phi\rangle = \frac{7}{9} - \frac{2}{9}\,\frac{\epsilon_p}{\sqrt{\epsilon_p^2 + 9\Delta^2}} \quad \text{for} \quad \epsilon_p \leq 0,$$

$$\langle\Phi|c_{iap}^\dagger c_{iap}|\Phi\rangle = \frac{2}{9} - \frac{2}{9}\,\frac{\epsilon_p}{\sqrt{\epsilon_p^2 + 9\Delta^2}} \quad \text{for} \quad \epsilon_p > 0.$$

In Fig. 5, these occupation numbers are presented.

For the color symmetric flavor asymmetric phase, if $i \in \{1,2,3\}, a \in \{u,d\}$, we find:

$$\langle\Phi|c_{iap}^\dagger c_{iap}|\Phi\rangle = \frac{2}{3} - \frac{1}{3}\,\frac{\epsilon_p}{\sqrt{\epsilon_p^2 + 3\Delta^2}} \quad \text{for} \quad \epsilon_p \leq 0,$$

$$\langle\Phi|c_{iap}^\dagger c_{iap}|\Phi\rangle = \frac{1}{3} - \frac{1}{3}\,\frac{\epsilon_p}{\sqrt{\epsilon_p^2 + 3\Delta^2}} \quad \text{for} \quad \epsilon_p > 0,$$

while

$$\langle\Phi|c_{isp}^\dagger c_{isp}|\Phi\rangle = 1 \quad \text{for} \quad \epsilon_p \leq 0, \quad \langle\Phi|c_{isp}^\dagger c_{isp}|\Phi\rangle = 0 \quad \text{for} \quad \epsilon_p > 0, \quad i \in \{1,2,3\}.$$

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For the color-flavor asymmetric phase, if $i \in \{1, 2\}, a \in \{u, d\}$, we find:

$$\langle \Phi | c^{\dagger}_{iap} c_{iap} | \Phi \rangle = \frac{1}{2} - \frac{1}{2} \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + \Delta^2}} \quad \text{for} \quad \epsilon_p \leq 0, \quad \text{and for} \quad \epsilon_p > 0,$$

while,

$$\langle \Phi | c^{\dagger}_{isp} c_{isp} | \Phi \rangle = 1 \quad \text{for} \quad \epsilon_p \leq 0, \quad \langle \Phi | c^{\dagger}_{isp} c_{isp} | \Phi \rangle = 0 \quad \text{for} \quad \epsilon_p > 0, \quad i \in \{1, 2, 3\}$$

$$\langle \Phi | c^{\dagger}_{3ap} c_{3ap} | \Phi \rangle = 1 \quad \text{for} \quad \epsilon_p \leq 0, \quad \langle \Phi | c^{\dagger}_{3ap} c_{3ap} | \Phi \rangle = 0 \quad \text{for} \quad \epsilon_p > 0, \quad a \in \{u, d, s\}.$$

For the CFL phase, if $(ia) \in \{(1u), (2d), (3s)\}$, we find:

$$\langle \Phi | c^{\dagger}_{iap} c_{iap} | \Phi \rangle = \frac{1}{2} - \frac{1}{6} \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + 4\Delta^2}} - \frac{1}{3} \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + \Delta^2}} \quad \text{for} \quad \epsilon_p \leq 0, \quad \text{and for} \quad \epsilon_p \leq 0,$$

while, if $(ia) \notin \{(1u), (2d), (3s)\}$, we have:

$$\langle \Phi | c^{\dagger}_{iap} c_{iap} | \Phi \rangle = \frac{1}{2} \left(1 - \frac{\epsilon_p}{\sqrt{\epsilon_p^2 + \Delta^2}} \right) \quad \text{for} \quad \epsilon_p \leq 0 \quad \text{and for} \quad \epsilon_p > 0.$$

In Fig. 6, these occupation numbers are presented.

5   Discussion and conclusions

We investigate, in a simple schematic model of the NJL type, the equilibrium between the 2CS pase, which is described by the color-flavor symmetric phase, and the CFL phase of strange quark matter at high densities. We focus on densities for which chiral symmetry is expected to have been restored. In the color-flavor symmetric phase, the quark numbers are independent of the color-flavor combination. In the CFL phase the quark numbers depend on the color-flavor combination, that is, the number of quarks associated with the color-flavor combinations ur, dg, sb is different from the number of quarks associated with the color flavor combinations ug, ub, dr, db, sr, sg. Our results were obtained for the coupling constant $G_c = 5 \text{GeV}^{-2}$ and are summarized in figs. 1, 2, 3 and 4. It is shown by fig. 1 that the color-flavor symmetric phase is stable for a chemical potential μ below $\mu_c = 0.505$ GeV, while the CFL phase is stable above, the equilibrium pressure being $P_c = 0.003$ GeV^4. In other words, the color-flavor symmetrical phase is stable for densities below the critical quark density $\rho_1 = 0.022$ GeV^3=2.75 fm^{-3}, which is 5.39 times the nuclear saturation density. The CFL phase is stable above the critical density $\rho_2 = 0.025$ GeV^3=3.125 fm^{-3} which is 6.13 times the nuclear saturation density. Between ρ_1 and ρ_2 both phases co-exist. This is illustrated by fig. 2, which also shows that the CFL phase leads to a considerably higher incompressibility than the one corresponding to the unphysical continuation of the color flavor symmetric curve. We have used a 3-momentum regularizing cutoff $\Lambda = 0.8$ GeV. As is shown by the tails of the occupation factors (see figs. 5, 6), this cutoff is adequate if the relevant chemical potential does not exceed 0.6 GeV. The chemical potential μ_c and the density ρ_2 are high enough to qualitatively support the approximation of neglecting the current mass of the strange quark. However, it would be interesting to investigate within the framework of the present approach the equilibrium between the color symmetric flavor asymmetric phase and the CFL phase, taking into account the strange quark mass. The present results are consistent with the hypothesis advanced in [5] that the interior of neutron stars may be made of strange matter. This calculation has been performed for the temperature $T = 0$. The 2SC phase which is described in the literature (see, e.g.

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[10]) differs from the color-flavor asymmetric phase in that the Lagrange multiplier μ_8 is used to implement the constraint $\langle \lambda_8 \rangle = 0$. The present work shows that this procedure should not be followed, since the color-flavor asymmetric phase is unitarily similar to the color-flavor symmetric phase and to the color symmetric flavor asymmetric phase and by implementing the constraint $\langle \lambda_8 \rangle = 0$ the thermodynamical potential is increased and the pressure is lowered so that the stability is reduced. Avoiding the constraint $\langle \lambda_8 \rangle = 0$ will have an effect on the critical point which is not irrelevant because, as fig. 1 shows, the P, μ curves have very close slopes.

It is important to take finite temperatures into account and we may ask which phase prevails at high temperature: CFL or the color-flavor symmetric one?

Acknowledgements

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